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THE STABILITY OF THE STEADY MOTIONS OF NON-HOLONOMIC MECHANICAL SYSTEMS WITH CYCLIC COORDINATES†

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The stability of the steady motions of non-holonomic mechanical systems of general form is investigated, on the assumption that they have cyclic coordinates and are subject to potential and dissipative forces. A stability theorem, generalizing a theorem proved previously in [1], is established. The problem of the stability of the steady motion of a three-wheeled carriage is considered as an example. © 2004 Elsevier Ltd. All rights reserved.

1. STEADY MOTIONS

Consider a non-holonomic mechanical system whose position is defined by generalized coordinates $q_1, \dots, q_n$. The velocities $\dot{q}_1, \dots, \dot{q}_n$ are constrained by $n-l$ $(l < n)$ time-independent non-holonomic relations

$$\dot{q}_\chi = \sum_{r=1}^{l} b_{\chi r}(q)\dot{q}_r \tag{1.1}$$

Here and below, the subscripts take the following values: $i = 1, \dots, k; j = 1, \dots, n; p, r, s = 1, \dots, l$; $\alpha, \beta, \gamma = k+1, \dots, l$; $\mu = m+1, \dots, n$; $\rho = l+1, \dots, m$; $\chi = l+1, \dots, n$.

We shall assume that the system is subject to potential forces (derivatives of a force function U) and dissipative forces (derivatives of a Rayleigh function F).

The equations of motion of a non-holonomic mechanical system, in the form of Voronets equations, have the following form [2, 3]

$$\frac{d}{dt}\frac{\partial\theta}{\partial\dot{q}_r} - \frac{\partial(\theta+U)}{\partial q_r} - \sum_{\chi=l+1}^{n}\frac{\partial(\theta+U)}{\partial q_\chi}b_{\chi r} - \sum_{\chi=l+1}^{n}\theta_\chi\sum_{s=1}^{l}\nu_{\chi rs}\dot{q}_s + \frac{\partial\Phi}{\partial\dot{q}_r} = 0 \tag{1.2}$$

where

$$\nu_{\chi rs} = \frac{\partial b_{\chi r}}{\partial q_s} - \frac{\partial b_{\chi s}}{\partial q_r} - \sum_{\chi'=l+1}^{n}\left(b_{\chi' r}\frac{\partial b_{\chi s}}{\partial q_{\chi'}} - b_{\chi' s}\frac{\partial b_{\chi r}}{\partial q_{\chi'}}\right)$$

Here θ, θ_χ and Φ are the results of eliminating the quantities $\dot{q}_\chi$ using constraints (1.1) from the expressions for T, $\partial T/\partial\dot{q}_\chi$, F, where T is the kinetic energy of the system,

$$2\theta = \sum_{r,s=1}^{l} a_{rs}(q)\dot{q}_r\dot{q}_s > 0, \quad \theta_\chi = \sum_{p=1}^{l}\theta_{\chi p}(q)\dot{q}_p, \quad 2\Phi = \sum_{r,s=1}^{l} f_{rs}(q_s)\dot{q}_s\dot{q}_r$$

Equations (1.2), together with Eqs (1.1), form a closed system of order $n+l$ in $q_j, \dot{q}_r$.

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Let us assume that the following conditions hold [1, 3]

$$\frac{\partial(T+U)}{\partial q_\mu} = 0, \quad \frac{\partial F}{\partial q_\mu} = 0, \quad \frac{\partial b_{\chi r}}{\partial q_\mu} = 0 \tag{1.3}$$

$$\frac{\partial(\theta+U)}{\partial q_\alpha} = 0, \quad \frac{\partial b_{\rho r}}{\partial q_\alpha} = 0, \quad \frac{\partial}{\partial q_\alpha}\sum_{\chi=l+1}^{n}\theta_{\chi p}\nu_{\chi rs} = 0, \quad \frac{\partial \Phi}{\partial q_\alpha} = 0 \tag{1.4}$$

Conditions (1.3) mean that the last $n - m$ equations of the non-holonomic constraints (1.1) are constraints of the Chaplygin type, and Eqs (1.2) may be considered independently of these constraints (the first $m - l$ constraints are constraints of general form). Conditions (1.4) mean that the coordinates q_α are cyclic in the sense of the definition of [1, 3]; the other coordinates q_i, q_p are positional.

Suppose that, under certain initial conditions, the system may have steady motions (SMs) in which the positional coordinates and cyclic coordinates are constant

$$q_i(t) = q_{i0}, \quad \dot{q}_i(t) = 0, \quad \dot{q}_\alpha(t) = \dot{q}_{\alpha 0} = \omega_\alpha, \quad q_\rho(t) = q_{\rho 0} \tag{1.5}$$

A necessary condition for the existence of SM (1.5) is that there must be no dissipation relative to the cyclic velocities, that is

$$\partial\Phi/\partial\dot{q}_\alpha = 0$$

When that is the case, the m constant quantities q_{i0}, $\bar{\omega}_\alpha$, $q_{\rho 0}$ satisfy the m equations

$$\begin{aligned}&\left(\frac{\partial U}{\partial q_i}\right)_0 + \sum_{\rho=l+1}^{m}\left(\frac{\partial U}{\partial q_\rho}b_{\rho i}\right)_0 + \\ &+\sum_{\gamma,\beta=k+1}^{l}\left\{\frac{1}{2}\left(\frac{\partial a_{\gamma\beta}}{\partial q_i} + \sum_{\rho=l+1}^{m}\frac{\partial a_{\gamma\beta}}{\partial q_\rho}b_{\rho i}\right)_0 + \sum_{\chi=l+1}^{n}\theta_{\chi\gamma}\nu_{\chi i\beta} - \sum_{\rho=l+1}^{m}\frac{\partial a_{i\beta}}{\partial q_\rho}b_{\rho\gamma}\right\}\omega_\gamma\omega_\beta = 0\end{aligned} \tag{1.6}$$

$$\begin{aligned}&\sum_{\gamma,\beta=k+1}^{l}\left\{\sum_{\chi=l+1}^{n}\theta_{\chi\gamma}\nu_{\chi\alpha\beta} + \sum_{\rho=l+1}^{m}\left[\frac{1}{2}b_{\rho\alpha}\frac{\partial a_{\gamma\beta}}{\partial q_\rho} - b_{\rho\gamma}\frac{\partial a_{\alpha\beta}}{\partial q_\rho}\right]\right\}_0\omega_\gamma\omega_\beta + \\ &+\sum_{\rho=l+1}^{m}\left\{b_{\rho\alpha}\frac{\partial U}{\partial q_\rho}\right\}_0 = 0\end{aligned} \tag{1.7}$$

$$\sum_{\alpha=k+1}^{l}(b_{\rho\alpha})_0\omega_\alpha = 0 \tag{1.8}$$

The zero subscript means that the expression is evaluated at values of the variables corresponding to SM (1.5).

It was pointed out [1, 3, 4] that in the general case system (1.6)–(1.8) has only trivial solutions for ω_α corresponding to equilibrium positions of the system. In some cases m_1 ($m_1 < m$) of Eqs (1.6)–(1.8) may turn out to be independent. Then the system may have a family of SMs of type (1.5), of dimension $m - m_1$.

Under conditions similar to those described in [5]

$$\sum_{\mu=m+1}^{n}(\theta_{\mu\beta}\nu_{\mu\alpha\gamma})_0 = -\sum_{\mu=m+1}^{n}(\theta_{\mu\gamma}\nu_{\mu\alpha\beta})_0, \quad (b_{\rho\alpha})_0 = 0 \tag{1.9}$$

Eqs (1.7) and (1.8) are satisfied for any ω_α, and in the system a manifold of SMs exists, the dimension of which is not less than the sum of the number of cyclic coordinates $(l - k)$ and the number of non-holonomic constraints of general form $(m - l)$.

In what follows we will assume that conditions (1.9) are satisfied. Then the system has an $(m-k)$-dimensional manifold of SMs, whose parameters $(q_{i0}, q_{\rho 0}, \omega_0)$ satisfy the system of equations

$$\left(\frac{\partial U}{\partial q_i}\right)_0 + \sum_{\rho=l+1}^{m}\left(\frac{\partial U}{\partial q_\rho}b_{\rho i}\right)_0 + \\ + \sum_{\alpha,\beta=k+1}^{l}\left[\frac{1}{2}\left(\frac{\partial a_{\alpha\beta}}{\partial q_i} + \sum_{\rho=l+1}^{m}\frac{\partial a_{\alpha\beta}}{\partial q_\rho}b_{\rho i}\right) + \sum_{\chi=l+1}^{n}\theta_{\chi\alpha}\nu_{\chi i\beta}\right]_0 \omega_\alpha\omega_\beta = 0 \qquad (1.10)$$

Let us discuss conditions (1.9). These conditions are satisfied, in particular, if

$$\sum_{\mu=m+1}^{n}(\theta_{\mu\beta}\nu_{\mu\alpha\gamma})_0 = 0, \quad (b_{\rho\alpha})_0 = 0 \qquad (1.11)$$

Obviously, a sufficient condition for conditions (1.11) to hold is [1, 3, 4, 6]

$$\sum_{\mu=m+1}^{n}\theta_{\mu\beta}\nu_{\mu\alpha\gamma} \equiv 0, \quad b_{\rho\alpha} \equiv 0 \qquad (1.12)$$

Note that conditions (1.12) will hold identically with respect to the positional coordinates, but conditions (1.11) hold only for SMs.

As already noted [5], previous investigations of the stability of SMs of non-holonomic mechanical systems [1, 3, 4, 6] have always assumed the truth of conditions (1.12), and these conditions are indeed satisfied in well-known problems of the SMs of a heavy rigid body (a disk, torus, etc.) on an absolutely rough horizontal plane, and in the problem of the motion of a "roller racer" [6]. In many problems, however, including the problem of the stability of the SM of a monocycle [5, 7–9], the conditions

$$\sum_{\mu=m+1}^{n}\theta_{\mu\beta}\nu_{\mu\alpha\gamma} \equiv 0$$

fail to hold, but instead one has

$$\sum_{\mu=m+1}^{n}(\theta_{\mu\beta}\nu_{\mu\alpha\gamma})_0 = 0$$

In the problem presented below, concerning the SMs of a three-wheeled carriage, which is a non-holonomic system with constraints of general form, conditions (1.11) hold but conditions (1.12) do not.

2. INVESTIGATION OF STABILITY

We choose a point of the manifold of SMs defined by formulae (1.10) and consider the question of whether a solution (1.5) of the system of equations (1.1) and (1.2) is stable to perturbations of the variables $q_i, \dot{q}_i, \dot{q}_\alpha, q_\rho$.

We introduce the differences

$$x_i = q_i - q_{i0}, \quad y_\alpha = \dot{q}_\alpha - \omega_\alpha, \quad z_\rho = q_\rho - q_{\rho 0}$$

The equations of perturbed motion, when conditions (1.9) are satisfied, in terms of the variables $x(k\times 1), y((l-k)\times 1), z((m-l)\times 1)$, have the form

$$\begin{aligned} A\ddot{x} + C\dot{y} &= W_1x + D_1\dot{x} + P_1y + V_1z + X(x, \dot{x}, y, z) \\ C^T\ddot{x} + B\dot{y} &= W_2x + D_2\dot{x} + V_2z + Y(x, \dot{x}, y, z) \\ \dot{z} &= W_3x + D_3\dot{x} + V_3z + Z(x, \dot{x}, y, z) \end{aligned} \qquad (2.1)$$

The formulae for the elements of the matrices A, C, ... are similar to the corresponding formulae of [3]; X, Y and Z are vector functions with terms of order higher than one in the variables just introduced.

Provided certain conditions are satisfied, the structure of the equations of perturbed motion (2.1) can be simplified considerably. For example, if conditions (1.12) are satisfied, the matrices W_2, V_2, W_3 and V_3 in Eqs (2.1) are zero matrices, that is, the equations corresponding to the cyclic velocities and the equations corresponding to the equations of the non-holonomic constraints do not contain terms that are linear in the variables x_i, y_α and z_ρ, and these equations obviously admit of $m-k$ linear integrals, to which there correspond $m-k$ zero roots of the characteristic equation of system (2.1) [1, 3]. The existence of the $(m-k)$-dimensional manifold of SMs implies that the linearized system (2.1) has $m-k$ zero roots, even when the matrices W_2, V_2, W_3 and V_3 do not vanish but satisfy the following conditions

$$W_2W_1^{-1}P_1=0,\quad W_3W_1^{-1}P_1=0,\quad V_2=W_2W_1^{-1}V_1,\quad V_3=W_3W_1^{-1}V_1\quad (\det W_1\neq 0) \tag{2.2}$$

We shall show that for non-holonomic systems with constraints of general form one can prove a theorem similar to the stability theorem proved in [5] for the SMs of Chaplygin systems.

It is not difficult to show (see [5]) that if $\det W_1\neq 0$ the change of variables

$$\eta = B_0y+C_0^T\dot{x}-D_{21}\chi,\quad \zeta = z-W_3W_1^{-1}A\dot{x}-W_3W_1^{-1}Cy-D_{31}x \tag{2.3}$$

where

$$B_0=B-W_2W_1^{-1}C,\ C_0^T=C^T-W_2W_1^{-1}A,\ D_{21}=D_2-W_2W_1^{-1}D_1,\ D_{31}=D_3-W_3W_1^{-1}D_1$$

and

$$\det B_0\neq 0 \tag{2.4}$$

brings system (2.1) to the form

$$\begin{aligned} &A_0\ddot{x}+D_0\dot{x}+W_0x+P_0\eta+V_0\zeta = X_0(x,\dot{x},\eta,\zeta)\\ &\dot{\eta} = Y_0(x,\dot{x},\eta,\zeta),\quad \dot{\zeta} = Z_0(x,\dot{x},\eta,\zeta)\end{aligned} \tag{2.5}$$

where

$$\begin{aligned} &A_0 = A-CB_0^{-1}C_0^T,\quad D_0 = -D_1-P_0C_0^T+CB_0^{-1}D_{21}-V_1W_3W_1^{-1}A\\ &W_0 = -W_1-V_1D_{31}+P_0D_{21},\quad P_0 = -(P_1+V_1W_3W_1^{-1}C)B_0^{-1},\quad V_0 = -V_1\end{aligned}$$

The functions $X_0(x,\dot{x},\eta,\zeta)$, $Y_0(x,\dot{x},\eta,\zeta)$, $Z_0(x,\dot{x},\eta,\zeta)$ are obtained from the functions $X(x,\dot{x},\eta,z)$, $Y(x,\dot{x},\eta,z)$, $Z(x,\dot{x},\eta,z)$ by applying the change of variables (2.3).

The characteristic equation corresponding to the linearized system (2.5) will obviously always have $m-k$ zero roots, while the other roots satisfy the equation

$$\det(A_0\lambda^2+D_0\lambda+W_0) = 0 \tag{2.6}$$

If some of the roots of Eq. (2.6) have positive real parts, then the SM (1.5) is unstable by Lyapunov's theorem of stability in the first approximation. Since under the conditions indicated the number of zero roots is identical with the dimension of the manifold of SMs (1.5) (as in the case considered in [1]), it follows that if all roots of Eq. (2.6) have negative real parts, we have the singular critical case of several zero roots, and the Lyapunov–Malkin theorem holds [10, 11].

We thus have a proposition analogous to a theorem of A. V. Karapetyan [1].

Theorem. A SM (1.5) of a non-holonomic system (1.1), (1.2) which has a manifold of SMs of dimension equal to the sum of the number of cyclic coordinates and the number of non-holonomic constraints of general form is stable (unstable) if all the roots of Eq. (2.6) have negative real roots (at least one root with positive real part). In the stable case, any perturbed motion sufficiently close to the unperturbed motion will tend to one of the possible SMs in the manifold (1.10) as $t\to\infty$.

It is important to note that condition (2.4) is essential, as the following example shows.

Example. Consider the classical problem of the motion of a Chaplygin sleigh on an inclined plane [2, 3]. A heavy rigid body rests on an inclined plane P on three feet, two of which are absolutely smooth, while the third is equipped with a semi-circular blade; the centre of mass of the body projects onto a point in the plane P on the straight line perpendicular to the blade and passing through the point K at which the blade is in contact with the plane P. The generalized coordinates are Cartesian coordinates ξ_1 and ξ_2 (the ξ_1 axis is parallel to the horizontal plane and the ξ_2 axis points upward with respect to the supporting plane P) of the point K and the angle φ of rotation of the body about a straight line perpendicular to the plane P. A non-holonomic constraint, representing the condition that the body will not slip at right angles to the plane of the blade, is described by the equation

$$\dot{\xi}_2 = \dot{\xi}_1 \operatorname{tg}\varphi \tag{2.7}$$

The Lagrangian has the form [2]

$$L = \frac{m}{2}[(\dot{\xi}_1 + l\dot{\varphi}\cos\varphi)^2 + (\dot{\xi}_2 + l\dot{\varphi}\sin\varphi)^2 + b^2\dot{\varphi}^2] - mg\sin\alpha(\xi_2 - l\cos\varphi)$$

where m is the mass, b is the radius of inertia, α is the angle of inclination of the plane and l is the distance from the projection of the centre of mass on the plane P to the point K. The ξ_1 coordinate is cyclic and ξ_2 and φ are positional coordinates. It is assumed that the system is subject to dissipative forces with Rayleigh function $F = mh\dot{\varphi}^2/2$. As remarked in [3], Eq. (2.7) does not describe a Chaplygin constraint.

The equations of motion in Voronets form (1.2) are

$$\begin{aligned} &\rho^2\ddot{\varphi} + \frac{l}{\cos\varphi}\ddot{\xi}_1 + \frac{l\sin\varphi}{\cos^2\varphi}\dot{\xi}_1\dot{\varphi} + h\dot{\varphi} + \delta l\sin\varphi = 0 \\ &\frac{l}{\cos\varphi}\ddot{\varphi} + \frac{1}{\cos^2\varphi}\ddot{\xi}_1 + \frac{\sin\varphi}{\cos^3\varphi}\dot{\xi}_1\dot{\varphi} + \delta\operatorname{tg}\varphi = 0 \end{aligned} \tag{2.8}$$

where

$$\rho^2 = b^2 + l^2, \quad \delta = g\sin\alpha$$

Equations (2.8), together with the constraint equation (2.7), constitute a closed system in the variables ξ_1, ξ_2 and φ.

It is not difficult to see that these equations admit of SMs of the form

$$\varphi(t) = \varphi_0, \quad (\varphi_0 = 0, \pi), \quad \dot{\varphi}(t) = 0, \quad \dot{\xi}_1 = \upsilon_0, \quad \xi_2 = \xi_{20} \tag{2.9}$$

which belong to a two-dimensional manifold and define uniform linear motion of the body at an arbitrary velocity υ_0, with the blade moving parallel to the ξ_1 axis. Note that in that case condition (1.11) is satisfied, but not condition (1.12) ($b_{\rho\alpha}(\varphi) = \operatorname{tg}\varphi \neq 0$, $b_{\rho\alpha}(\varphi_0) = 0$).

Equations (2.8) and (2.7) corresponding to (2.1), linearized in the neighbourhood of the SM (2.9), have the form

$$\rho^2\ddot{x} + l\varepsilon\dot{y} = -h\dot{x} - l\varepsilon\delta x, \quad l\varepsilon\ddot{x} + \dot{y} = -\delta x, \quad \dot{z} = \upsilon_0 x \tag{2.10}$$

where

$$x = \varphi - \varphi_0, \quad y = \dot{\xi}_1 - \upsilon_0, \quad z = \xi_2 - \xi_{20}, \quad \varepsilon = \cos\varphi_0 = \pm 1$$

In the notation we have adopted, the matrices of the system are

$$A = \rho^2, \quad C = \varepsilon l, \quad B = 1, \quad D_1 = 0, \quad D_2 = -h_1, \quad D_3 = 0$$

$$P_1 = P_2 = 0, \quad V_1 = V_2 = V_3 = 0, \quad W_1 = -\varepsilon\delta l, \quad W_2 = -\delta, \quad W_3 = \upsilon_0$$

Hence it follows that $B_0 = 0$ and condition (2.4) does not hold, so that the theorem formulated above is not applicable.

In the system under consideration, the dimension of the manifold of SMs is two, but the characteristic equation of system (2.10) has three zero roots. This system belongs to the usual critical case (in Lyapunov's sense) of several zero roots.

It is not difficult to see that the SM (2.9) (including the equilibrium position) is unstable, irrespective of the presence of dissipative forces with respect to the coordinate φ. Indeed, subtract the second equation of system (2.8), multiplied by $l\cos\varphi$, from the first. System (2.8) then becomes

$$b^2\ddot{\varphi} + h\dot{\varphi} = 0, \quad \ddot{\xi}_1 + \dot{\xi}_1\dot{\varphi}\,\mathrm{tg}\varphi = \frac{hl}{b^2}\dot{\varphi}\cos\varphi - \frac{1}{2}\delta\sin 2\varphi$$

The corresponding equations of perturbed motion become

$$b^2\ddot{x} + h\dot{x} = 0, \quad \dot{y} + \gamma(t)y = \psi(t) \tag{2.11}$$

where

$$\gamma(t) = \dot{x}\,\mathrm{tg}(\varphi_0 + x)$$

$$\psi(t) = \frac{hl}{b^2}\dot{x}\cos(\varphi_0 + x) - \dot{x}\upsilon_0\,\mathrm{tg}(\varphi_0 + x) - \frac{\delta}{2}\sin 2(\varphi_0 + x)$$

By the first equation of system (2.11) we have

$$x(t) = C_1 + C_2\exp\left(-\frac{h}{b^2}t\right)$$

where C_1 and C_2 are arbitrary constants.

We now choose the following initial data for system (2.11)

$$t = 0{:}\ x(0) = x_0 \neq 0, \frac{\pi}{2}, \quad \dot{x}(0) = 0, \quad (C_1 = x_0, C_2 = 0), \quad y(0) = 0$$

Then

$$x(t) \equiv x_0, \quad \dot{x}(t) \equiv 0, \quad \gamma(t) \equiv 0, \quad \psi(t) = -\frac{\delta}{2}\sin 2x_0; \quad y(t) = -\left(\frac{\delta}{2}\sin 2x_0\right)t \to \infty \quad \text{as} \quad t \to \infty$$

Note that in the case when dissipative forces are acting with respect to all the coordinates ξ_1, ξ_2 and φ with Rayleigh function

$$F = \frac{m}{2}[h\dot{\varphi}^2 + h_1(\dot{\xi}_1^2 + \dot{\xi}_2^2)]$$

the SM (2.9) are only equilibrium positions ($\upsilon_0 = 0$), of which, under certain conditions, the equilibrium position $\varphi = 0$ is stable and $\varphi = \pi$ is unstable [2, 3].

3. STEADY MOTIONS OF A THREE-WHEELED CARRIAGE

Let us consider the problem of the SMs of a three-wheeled carriage moving on an absolutely rough horizontal plane. Ignoring the inertia of the revolving wheels, we can represent a simplified model of the carriage (a tricycle) by a system of two rigid bodies [12]: a body of mass m_1, consisting of a body and a rigidly attached axis fitted with two wheels, and a body of mass m_2, which is a vertical post with a front wheel. A special case of this problem (the problem of the motion of a "roller racer") was considered in [6].

The position of the system is defined by the coordinates x, y, θ and ψ (see Fig. 1): x and y are the coordinates of the point O – the midpoint of the rear bridge in a fixed system of coordinates $\bar{O}xy$, ψ is the angle between the axis of symmetry Ox_1 of the carriage and the fixed axis $\bar{O}x$, A is the projection of the point at which the post is mounted on the Ox_1 axis, θ is the angle defining the position of the Ax_2 axis of the front part of the tricycle relative to the Ox_1 axis, B is the projection of the centre of the front wheel on the xy plane, and C_1 and C_2 are the projections of the centres of mass of the first and second bodies on the Ox_1 and Ax_1 axes, respectively. Let

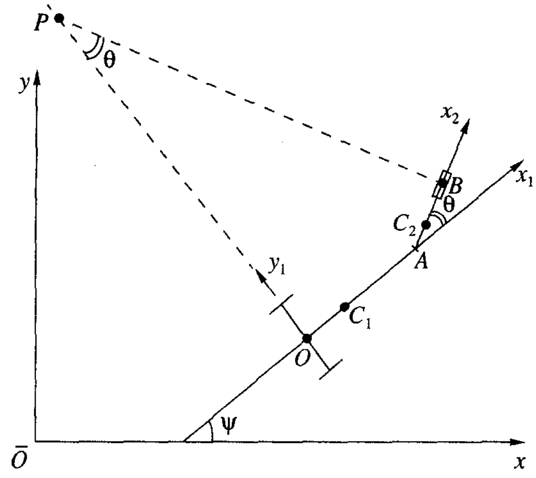

Fig. 1

$$b = AB, \quad l = OA, \quad l_1 = OC_1, \quad d = AC_2$$

(We are ignoring the displacement of the centre of mass of the tricycle due to rotation of the rear part through the angle θ.)

The equations of the non-holonomic constraints, expressing the conditions that the points O and B have zero components of the velocity in the transverse direction are

$$\begin{aligned} &-\dot{x}\sin\psi + \dot{y}\cos\psi = 0 \\ &-\dot{x}\sin(\psi+\theta) + \dot{y}\cos(\psi+\theta) + l\dot{\psi}\cos\theta + b(\dot{\psi}+\dot{\theta}) = 0 \end{aligned} \tag{3.1}$$

Without loss of generality, we can assume that $\sin\psi \neq 0$ and solve Eqs (3.1) for $\dot{x}$, $\dot{\psi}$:

$$\dot{x} = \dot{y}\,\mathrm{ctg}\,\psi, \quad \dot{\psi} = \frac{1}{r}\left[\frac{\sin\theta}{\sin\psi}\dot{y} - b\dot{\theta}\right]; \quad r = b + l\cos\theta \tag{3.2}$$

Remark. When the procedure used in [6, 12] to eliminate $\dot{x}$ and $\dot{y}$ from Eq. (3.1) is applied, it is assumed that $\sin\theta \neq 0$. This assumption in fact excludes the possibility of investigating the simplest motion of the tricycle – along a straight line at constant velocity υ_0 (since in that case $\theta = 0$ or $\theta = \pi$). In addition, the derivation in [12] of the equations linearized in the neighbourhood of rectilinear motion involves an error.

The kinetic energy may be written, assuming the validity of the constraints (3.2), as

$$\begin{aligned} \Theta = \frac{1}{2r^2}\Bigg\{&[(m_1+m_2)r^2 + (I_1+I_2)\sin^2\theta]\frac{\dot{y}^2}{\sin^2\psi} + \\ &+ [I_1 b^2 + I_2 l^2\cos^2\theta]\dot{\theta}^2 - 2(I_1 b - I_2 l\cos\theta)\frac{\sin\theta}{\sin\psi}\dot{\theta}\dot{y}\Bigg\} \end{aligned} \tag{3.3}$$

$$I_1 = I_{11} + m_1 l_1^2 + m_2 l^2, \quad I_2 = I_{22} - 2m_2 bd$$

where I_1 and I_2 are the reduced moments of inertia, I_{11} is the moment of inertia of the first body about a vertical axis passing through the point C_1 and I_{22} is the moment of inertia of the second body about a vertical axis passing through the point A.

The first equation of (3.2) represents a non-holonomic constraint of the Chaplygin type and the second, a non-holonomic constraint of the general type.

It is assumed that the system is subject to dissipative forces which are the derivatives of the Rayleigh function $f = h\dot{\theta}^2/2$.

We introduce a variable $\upsilon = \dot{y}/\sin\psi$ – the projection of the velocity of the point O on to the axis of symmetry of the carriage. Then the equations of motion of the system, set up on the basis of the Voronets equations, become

$$\begin{aligned}
\frac{d}{dt}[F_1(\theta)\dot{\theta} + F_2(\theta)\upsilon] &= R_1(\theta)\dot{\theta}^2 + R_2(\theta)\dot{\theta}\upsilon + R_3(\theta)\upsilon^2 - h\dot{\theta} \\
\frac{d}{dt}[F_3(\theta)\dot{\theta} + F_4(\theta)\upsilon] &= S_1(\theta)\dot{\theta}^2 + S_2(\theta)\dot{\theta}\upsilon \\
\dot{\psi} &= \frac{1}{r}(\upsilon\sin\theta - b\dot{\theta})
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}
F_1(\theta) &= \frac{1}{r^2}(I_1 b^2 + I_2 l^2\cos^2\theta) \\
F_2(\theta) &= F_3(\theta) = \frac{1}{r^2}(I_2 l\cos\theta - I_1 b)\sin\theta \\
F_4(\theta) &= m_1 + m_2 + \frac{1}{r^2}(I_1 + I_2)\sin^2\theta \\
R_1(\theta) &= -\frac{1}{r^3}(I_2 l\cos\theta - I_1 b)bl\sin\theta \\
R_2(\theta) &= -\left[\frac{1}{r^3}(I_1 + I_2)bl\sin^2\theta + m_2 dl^2 r\cos\theta + b^2 rM\right] \\
R_3(\theta) &= \frac{1}{r^2}(bM - m_2 dl)\sin\theta \\
S_1(\theta) &= \frac{1}{r^3}[(I_2 l\cos\theta - I_1 b)(l + b\cos\theta) + (Mb^2 + m_2 dl^2\cos\theta)r] \\
S_2(\theta) &= \frac{1}{r^3}[(I_1 + I_2)(l + b\cos\theta) + (m_2 dl - Mb)r]\sin\theta \\
M &= m_1 l_1 + m_2 l
\end{aligned}$$

The equations of motion of system (3.4) admit of particular solutions

$$\theta = \theta_0, \quad \dot{\theta} = 0, \quad \upsilon = \upsilon_0, \quad \psi_0 = \upsilon_0\sin\theta_0/r_0 \tag{3.5}$$

describing SMs.

The parameters θ_0 and υ_0 satisfy the condition

$$\upsilon_0(bM - m_2 dl)\sin\theta_0 = 0 \quad \text{or} \quad \upsilon_0[m_1 bl_1 + m_2 l(b-d)]\sin\theta_0 = 0 \tag{3.6}$$

(it is assumed that $r_0 = b + l\cos\theta_0 \neq 0$).

Note the fact that condition (3.6) corresponds to the first condition of (1.11), while no condition of the form (1.12) is satisfied.

Condition (3.6) will hold if

(1) $\sin\theta_0 = 0$, υ_0 is an arbitrary constant; the SM is rectilinear motion at constant velocity $\upsilon_0 \neq 0$ and an arbitrary angle $\psi_0 \neq 0, \pi$ to the x axis ($\upsilon_0 = \dot{y}_0/\sin\psi_0$);

(2) $\upsilon_0 = 0$; the SM is equilibrium (θ_0 is an arbitrary constant);

(3) $m_1 bl_1 + m_2 l(b - d) = 0$; in that case θ_0 and υ_0 are arbitrary constants ($\theta_0 \neq 0, \pi$); in particular, this condition is satisfied if $b = d$, $l_1 = 0$ or if $m_2 = 0$, $l_1 = 0$ [6]; the SM is rotation of the system about its instantaneous centre – the point P ($OP = r_0/\sin\theta_0$) (see Fig. 1).

Thus, a two-dimensional manifold of SMs exists, of dimension equal to the sum of the number of cyclic coordinates (y) and the number of non-holonomic constraints of general form, which are determined by the second relation of (3.2).

The equations linearized in the neighbourhood of the SM, in terms of perturbations

$$\xi = \theta - \theta_0, \quad \eta = \upsilon - \upsilon_0, \quad \zeta = \psi - \psi_0$$

have the form

$$A\ddot{\xi} + C\dot{\eta} = W_1\xi + D_1\dot{\xi} + P_1\eta, \quad C^T\ddot{\xi} + B\dot{\eta} = D_2\dot{\xi}, \quad \dot{\zeta} = W_3\xi + D_3\dot{\xi} + P_3\eta \tag{3.7}$$

where

$$A = F_1(\theta_0), \quad C = F_2(\theta_0), \quad W_1 = \left(\frac{dR_3}{d\theta}\right)_0 \upsilon_0^2, \quad D_1 = \left[R_2(\theta_0) - \left(\frac{dF_2}{d\theta}\right)_0\right]\upsilon_0 - h$$

$$P_1 = 2R_3(\theta_0)\upsilon_0, \quad B = F_4(\theta_0), \quad D_2 = \left[S_2(\theta_0) - \left(\frac{dF_4}{d\theta}\right)_0\right]\upsilon_0$$

$$W_3 = \frac{l + b\cos\theta_0}{r_0^2}\upsilon_0, \quad D_3 = -\frac{b}{r_0}, \quad P_3 = \frac{\sin\theta_0}{r_0}$$

The characteristic equation of system (3.7) has two zero roots, corresponding to the existence of the two-dimensional manifold of SMs; the other roots are determined from the equation

$$B(A\lambda^2 - D_1\lambda - W_1) - (C\lambda - P_1)(C^T\lambda - D_2) = 0$$

In case 1, ($\sin\theta_0 = 0$, $\varepsilon = \cos\theta_0 = \pm 1$) $C = 0$, $P_1 = 0$, $D_2 = 0$. Condition (2.4) is satisfied in this problem, since $B_0 = B \neq 0$. Under these conditions we have $A_0 = A$, $D_0 = -D_1$, $W_0 = W_1$ in Eq. (2.6) and, according to the theorem proved above, the conditions for the rectilinear motion (3.6) to be stable have the form $W_1 < 0$, $D_1 < 0$, that is

$$\varepsilon K_1 < 0, \quad hr_0^2 + \upsilon_0 K_2 > 0 \tag{3.8}$$

where

$$K_1 = m_1 l_1 b + m_2 l(b - d); \quad K_2 = I_{22} l - \varepsilon I_1 b + m_1 b^2 l_1 + m_2 l(b^2 - 2bd + \varepsilon dl)$$

This SM is unstable if either of inequalities (3.8) fails to hold.

Note that the quantity D_1 does not vanish and is a linear function of υ_0. This means that when there are no dissipative forces ($h = 0$), if the parameters of the system are such that $K_2 > 0$, the SM is stable when $\upsilon_0 > 0$, asymptotically stable with respect to part of the variables (θ, $\dot{\theta}$), and unstable when the carriage is moving in the opposite direction ($\upsilon_0 < 0$).

Thus, in this problem, as in the problem of the Celtic stone [3, 4, 13], both distinctive features of non-holonomic systems are clearly represented: asymptotic stability of a conservative system with respect to part of the variables, and dependence of the nature of the stability on the direction of motion.

It follows from the first condition of (3.8) that a necessary condition for the stability of rectilinear motion is $\theta = \pi$, since as a rule $K_1 > 0$. This means that the front wheel must be pulled "backwards" relative to the direction of motion.

Note that, if dissipation is present, it follows from the second condition of (3.8) that motion in the opposite direction ($\upsilon_0 < 0$) when $K_2 > 0$ may be stable if $\theta = 0$ ($\varepsilon = 1$) at moderate velocities of motion.

These conclusions regarding the stability of the rectilinear motion of the tricycle agree with the results in [14].

In case 3 we have $K_1 = 0$. Then $R_3 \equiv 0$, $P_1 = 0$, $W_1 = 0$. The condition for stability of the SM is

$$\frac{\upsilon_0}{r_0^3}\{(m_1 + m_2)[(I_2 l\cos\theta_0 - I_1 b)(l + b\cos\theta_0) + m_2 dlr_0^2] + (I_1 + I_2)m_2 dl\sin^2\theta_0\} +$$

$$+\left[(m_1 + m_2) + \frac{(I_1 + I_2)\sin^2\theta_0}{r_0^2}\right]h > 0$$

In particular, if $h = 0$ and $m_2 = 0$, we have

$$\frac{v_0(I_1 l\cos\theta_0 - I_1 b)(l + b\cos\theta_0)}{b + l\cos\theta_0} > 0$$

which agrees with the results in [6].

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